ON DOUBLE 3-TERM ARITHMETIC PROGRESSIONS

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Abstract

In this note we are interested in the problem of whether or not every increasing sequence of positive integers $x_1x_2x_3\ldots$ with bounded gaps must contain a *double* 3-term arithmetic progression, i.e., three terms x_i, x_j, and x_k such that $i+k=2j$ and $x_i+x_k=2x_j$. We consider a few variations of the problem, discuss some related properties of double arithmetic progressions, and present several results obtained by using `RamseyScript`, a high-level scripting language.

1 Introduction

In 1987, Tom Brown and Allen Freedman ended their paper titled *Arithmetic progressions in lacunary sets* [Brown and Freedman 1987] with the following conjecture.

Conjecture 1. *Let x_i be a sequence of positive integers with $1 \le x_i \le K$. Then there are two consecutive intervals I, J of the same length, with $\sum_{i\in I} x_i = \sum_{j\in J} x_j$. Equivalently, if $a_1 < a_2 < \cdots$ satisfy $a_{n+1} - a_n \le K$, for all n, then there exist $x < y < z$ such that $x+z=2y$ and $a_x + a_z = 2a_y$.*

If true, Conjecture 1 would imply that if the sum of the reciprocals of a set $A = \{a_1 < a_2 < a_3 < \cdots\}$ of positive integers diverges, and $a_{n+1} - a_n \to \infty$ as $n \to \infty$, and there exists K such that $a_{i+1} - a_i \le a_{j+1} - a_j + K$ for all $1 \le i \le j$,

then A contains a 3-term arithmetic progression. This is a special case of the famous Erdős conjecture that if the sum of the reciprocals of a set A of positive integers diverges, then A contains arbitrarily long arithmetic progressions.

Conjecture 1 is a well-known open question in the combinatorics of words and it is usually stated in the form:

> Must every infinite word on a finite alphabet consisting of positive integers contain an additive square, i.e., two adjacent blocks of the same length and the same sum?

The answer is trivially yes in case the alphabet has size at most 3. For more on this question see, for example, [Au et al. 2011, Freedman 2013+], [Grytczuk 2008]. Also see [Halbeisen and Hungerbühler 2000], [Pirillo and Varricchio 1994] and [Richomme et al. 2011].

We mention two relatively recent positive results. Freedman [Freedman 2013+] has shown that if $a < b < c < d$ satisfy the Sidon equation $a + d = b + c$, then every word on $\{a, b, c, d\}$ of length 61 contains an additive square. His proof is a clever reduction of the general problem to several cases that are then checked by computer.

Ardal, Brown, Jungić, and Sahasrabudhe [Ardal et al. 2012] proved that if an infinite word $\omega = a_1a_2a_3\ldots$ has the property that there is a constant M, such that for any positive integer n the number of possible sums of n consecutive terms in ω does not exceed M, then for any positive integer k there is a k term arithmetic progression $\{m + id : i = 0, \ldots, k - 1\}$ such that

$$\sum_{i=m+1}^{m+d} a_i = \sum_{i=m+d+1}^{m+2d} a_i = \ldots = \sum_{i=m+(k-2)d+1}^{m+(k-1)d} a_i.$$

The proof of this fact is based on van der Waerden's theorem [van der Waerden 1927].

This note is inspired by the second statement in Conjecture 1.

Problem 1. *Does every increasing sequence of positive integers*

$$a_1a_2a_3\ldots$$

with bounded gaps contain a double 3-term arithmetic progression, i.e, three terms a_x*,* a_y*, and* a_z *such that*

$$x + z = 2y \text{ and } a_x + a_z = 2a_y?$$

It is straightforward to check that Problem 1 is equivalent to the question above concerning additive squares: Given positive integers K and $a_1 < a_2 < a_3 < \cdots$,

with $a_{i+1} - a_i \leq K$ for all $i \geq 1$, let $x_i = a_{i+1} - a_i,\ i \geq 1$. Then $x_1x_2x_3\ldots$ is an infinite word on a finite alphabet of positive integers. Given an infinite word $x_1x_2x_3\ldots$ on a finite alphabet of positive integers, define $a_1, a_2, a_3, \cdots$ recursively by $a_1 \in \mathbb{N}, a_{i+1} = x_i + a_i,\ i \geq 1$. Then $a_1 < a_2 < a_3 \cdots$, and $a_{i+1} - a_i \leq K$ for some K and all $i \geq 1$. In both cases, an additive square in $x_1x_2x_3\ldots$ corresponds exactly to a double 3-term arithmetic progression in $a_1 < a_2 < a_3 < \cdots$.

The existence of an infinite word on four integers with no additive cubes, i.e., with no three consecutive blocks of the same length and the same sum, established by Cassaigne, Currie, Schaeffer, and Shallit [Cassaigne et al. 2013+], translates into the fact that there is an increasing sequence of positive integers with bounded gaps with no double 4-term arithmetic progression.

But what about a *double* variation on van der Waerden's theorem?

Problem 2. *If the set of positive integers is finitely coloured, must there exist a colour class, say $A = \{a_1 < a_2 < a_3 < \cdots\}$ for which there exist $i < j < k$ with $a_i + a_k = 2a_j$ and $i + k = 2j$?*

We have just seen that an affirmative answer to Problem 1 gives an affirmative answer to the question concerning additive squares. It is also true that an affirmative answer to Problem 1 implies an affirmative answer to Problem 2.

Proposition 1. *Assume that every increasing sequence of positive integers $x_1x_2x_3\ldots$ with bounded gaps contains a double 3-term arithmetic progression. Then if the set of positive integers is finitely coloured, there must exist a colour class, say $A = \{a_1 < a_2 < a_3 < \cdots\}$, which contains a double 3-term arithmetic progression.*

Proof. We use induction on the number of colours, denoted by r. For $r = 1$ the conclusion trivially follows. Now assume that for every r-colouring of $\mathbb{N}$ there exists a colour class which contains a double 3-term arithmetic progression. By standard methods (similar to those used in the proof of Theorem 4 below) there exists $M \in \mathbb{N}$ such that every r-colouring of $[1, M]$ (or of any translate of $[1, M]$) yields a monochromatic double 3-term arithmetic progression.

Assume now that there is an $(r + 1)$-colouring of $\mathbb{N}$ for which there does *not* exist a monochromatic double 3-term arithmetic progression. Let the $(r + 1)$st colour class be $C(r + 1) = \{x_1 < x_2 < \cdots\}$. By the induction hypothesis on r colours, $C(r + 1)$ is infinite. By the assumption that every increasing sequence of positive integers $x_1x_2x_3\ldots$ with bounded gaps contains a double 3-term arithmetic progression, $C(r + 1)$ does not have bounded gaps. In particular, there is $p \geq 1$ such that $x_{p+1} - x_p \geq M + 2$. But then the interval $[x_p + 1, x_{p+1} - 1]$ contains a translate of $[1, M]$ and is coloured with only r colours, so that $[x_p+1, x_{p+1}-1]$ does contain a monochromatic double 3-term arithmetic progression. This contradiction completes the proof. $\square$

More generally, if the set of positive integers is finitely coloured and if each colour class is regarded as an increasing sequence, must there be a monochromatic double k-term arithmetic progression, for a given positive integer k? What if the gaps between consecutive elements coloured with same colour are pre-prescribed, say at most 4 for the first colour, at most 6 for the second colour, and at most 8 for the third colour, and so on?

In the spirit of van der Waerden's numbers $w(r, k)$ [Graham et al. 1990] we define the following.

Definition 1. *For given positive integers r and k greater than 1, let $w^*(r, k)$ be the least integer, if it exists, such that for any r-colouring of the interval $[1, w^*(r, k)]$ there is a monochromatic double k-term arithmetic progression.*

For given positive numbers r, k, $a_1, a_2, \ldots, a_r$ let $w^*(k; a_1, a_2, \ldots, a_r)$ be the least integer, if it exists, such that for any r-colouring of the interval $[1, w^*(k; a_1, a_2, \ldots, a_r)] = A_1 \cup A_2 \cup \ldots \cup A_r$ such that for each i the gap between any two consecutive elements in A_i is not greater than a_i there is a monochromatic double k-term arithmetic progression.

We will show that $w^*(2, 3)$ is relatively simple to obtain. We will give lower bounds for $w^*(3, 3)$ and $w^*(4, 2)$ and a table with values of $w^*(3; a_1, a_2, a_3)$ for various triples (a_1, a_2, a_3) and propose a related conjecture.

We will share with the reader some insights related to the general question about the existence of double 3-term arithmetic progressions in increasing sequences with bounded gaps.

Finally, we will describe `RamseyScript`, a high-level scripting language developed by the third author that was used to obtain the colourings and bounds that we have established.

2 $w^*(r, 3)$

Now we look more closely at $w^*(r, 3)$, the least integer, if it exists, such that for every r-colouring of the interval $[1, w^*(r, 3)]$ there is a monochromatic double 3-term arithmetic progression. That is, for every r-colouring of $[1, w^*(r, 3)]$ there is a colour class $A = \{a_1 < a_2 < a_3 < \cdots\}$ and $i < j < k$ with $a_i + a_k = 2a_j$ and $i + k + 2j$.

Suppose that $w^*(r, 3)$ does not exist for some r, but $w^*(r - 1, 3)$ does exist. A standard argument then shows that there is a colouring of the positive integers with r colours, say with colour classes $A_1, A_2, \ldots, A_r$, such that no colour class contains a double 3-term arithmetic progression. Then (a) A_1 contains no double 3-term arithmetic progression, (b) A_1 has bounded gaps because $w^*(r-1, 3)$ exists, and (c) A_1 is infinite, because $w^*(r - 1, 3)$ exists.

Let $d_1, d_2, \ldots$ be the sequence of consecutive differences of the sequence A_1. That is, if $A_1 = \{a_1, a_2, a_3, \ldots\}$ then $d_n = a_n - a_{n-1}$, $n \geq 1$. Then the sequence $d_1, d_2, \ldots$ is a sequence on a finite set of integers which does not contain any additive square.

Thus if there exists r such that $w^*(r, 3)$ does not exist then there exists a sequence on a finite set of integers which does not contain an additive square.

It is conceivable that proving that $w^*(r, 3)$ does not exist for all r (if this is true!) is easier than proving directly the existence of a sequence on a finite set of integers with no additive square.

Theorem 1. $w^*(2, 3) = 17$.

Proof. Colour $[1, m]$ with two colours, with no monochromatic double 3-term arithmetic progressions. Then the first colour class must have gaps of either 1, 2, or 3. Thus the sequence of gaps of the first colour class is a sequence of 1s, 2s, and 3s, and this sequence must have length at most 7, otherwise there is an additive square, which would give a double 3-term arithmetic progression in the first colour class. Hence, the first colour class can contain at most 8 elements (only 7 consecutive differences) and similarly for the second colour class. This shows that $w^*(2, 3) \leq 8 + 8 + 1 = 17$. A short calculation, which can be done by hand, shows that $w^*(2, 3) = 17$. □

Theorem 2. $w^*(3, 3) \geq 414$.

The following 3-colouring of $[1, 413]$ avoids monochromatic 3-term double arithmetic progressions:

$A_1 = \{1, 3, 6, 10, 12, 13, 15, 18, 19, 21, 22, 29, 31, 32, 34, 37, 38, 40, 44, 51, 55, 66, 70, 71, 74, 75, 77, 80, 82, 87, 90, 91, 94, 96, 97, 99, 111, 112, 114, 115, 119, 124, 125, 128, 129, 131, 132, 135, 136, 140, 142, 143, 146, 148, 149, 156, 159, 160, 162, 163, 168, 170, 184, 186, 189, 191, 192, 195, 197, 200, 204, 206, 214, 215, 231, 232, 235, 236, 243, 244, 246, 249, 251, 252, 267, 268, 271, 272, 274, 277, 278, 281, 289, 290, 293, 295, 296, 301, 304, 305, 308, 310, 311, 316, 327, 328, 331, 335, 336, 339, 340, 342, 343, 346, 347, 354, 356, 357, 359, 362, 363, 365, 366, 371, 372, 374, 377, 378, 380, 387, 389, 392, 394, 395, 398, 402, 403, 408\}$,

$A_2 = \{2, 4, 5, 9, 11, 16, 20, 25, 28, 30, 33, 39, 41, 42, 45, 46, 48, 49, 54, 56, 57, 59, 62, 63, 69, 72, 73, 76, 83, 84, 98, 100, 103, 105, 106, 108, 113, 116, 117, 120, 122, 123, 126, 127, 138, 139, 144, 145, 147, 150, 151, 153, 154, 158, 164, 165, 167, 171, 172, 174, 175, 178, 179, 181, 185, 187, 188, 190, 193, 194, 196, 201, 207, 210, 211, 213, 217, 218, 221, 222, 224, 227, 228, 233, 234, 237, 238, 240, 253, 255, 258, 260, 261, 264, 269, 270, 273, 275, 276, 279, 283, 284, 286, 294, 297, 298, 312, 315, 319, 322, 323, 325, 326, 330, 332, 341, 344, 345, 350, 352, 353, 355, 358, 360, 361, 369, 370, 373, 375, 376, 379, 383, 385, 386, 401, 404, 405, 409, 410, 412, 413\}$,

$A_3 = \{7, 8, 14, 17, 23, 24, 26, 27, 35, 36, 43, 47, 50, 52, 53, 58, 60, 61, 64, 65, 67, 68, 78, 79, 81, 85, 86, 88, 89, 92, 93, 95, 101, 102, 104, 107, 109, 110, 118, 121, 130, 133, 134, 137,$

$141, 152, 155, 157, 161, 166, 169, 173, 176, 177, 180, 182, 183, 198, 199, 202, 203, 205, 208, 209, 212, 216, 219, 220, 223, 225, 226, 229, 230, 239, 241, 242, 245, 247, 248, 250, 254, 256, 257, 259, 262, 263, 265, 266, 280, 282, 285, 287, 288, 291, 292, 299, 300, 302, 303, 306, 307, 309, 313, 314, 317, 318, 320, 321, 324, 329, 333, 334, 337, 338, 348, 349, 351, 364, 367, 368, 381, 382, 384, 388, 390, 391, 393, 396, 397, 399, 400, 406, 407, 411\}$.

This colouring is the result of about 8 trillion iterations of `RamseyScript`, using the Western Canada Research Grid[1]. We started with a seed 3-colouring of the interval $[1, 61]$ and searched the entire space of extensions. Figure 1 gives the number of double 3-AP free extensions of the seed colouring versus their lengths.

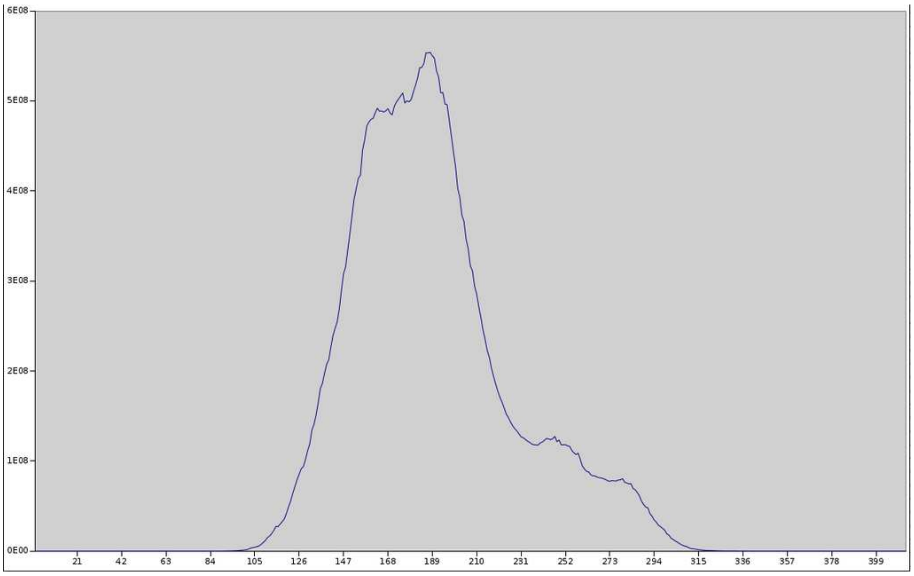

Figure 1: Number of double 3-AP free extensions versus length

To get more information about $w^*(3, 3)$ we define $w^*(3, 3; d)$ to be the smallest m such that whenever $[1, m]$ is 3-coloured so that each colour class has maximum gap at most d, then there is a monochromatic double 3-term arithmetic progression. Our goal was to compute $w^*(3; 3; d)$ for small values of d. (See Table 1.)

We note that $w^*(3, 3; d)$ is already difficult to compute when d is much smaller than $w^*(2, 3) = 17$. (In a 3-colouring containing no monochromatic double 3-term arithmetic progression the maximum gap size of any colour class is 17.)

Freedman [Freedman 2013+] showed that there were 16 double 3-AP free 51-term sequences having the maximum gap of at most 4. The fact that $w^*(3, 3; 4) = 39$ is an interesting contrast, and shows that the "stand-alone" sequences are at least somewhat less restricted.

Theorem 3. *$w^*(4, 2) \geq 30830$.*

[1] http://www.westgrid.ca

		$w^*(3,3;d)$
Max gap d	2	11
	3	22
	4	39
	5	100
	6	> 152
	7	?

Table 1: Known Values of $w^*(3,3;d)$

Starting with the seed 2-colouring $[1,10] = \{1,4,6,7\} \cup \{2,3,5,8,9,10\}$, after $2 \cdot 10^8$ iterations `RamseyScript` produced a double 4-AP free 2-colouring of the interval $[1,30829]$.

3 $w^*(3;a,b,c)$ and $w^*(k;a,b)$

Recall that $w^*(3;a,b,c)$ is the least number such that every 3-colouring of $[1,w^*(3;a,b,c)]$, with gap sizes on the three colours restricted to a, b, and c, respectively, has a monochromatic double 3-term arithmetic progression. Similarly, $w^*(k;a,b)$ is the least number such that every 2-colouring of $[1,w^*(k;a,b)]$, with gap sizes on the two colours restricted to a and b, respectively, has a monochromatic double k-term arithmetic progression.

Table 2 shows values of $w^*(3;a,b,c)$ for some small values of a, b, and c. Table 3 shows values of $w^*(k;a,b)$ for some small values of a, b, and k.

Based on this evidence, we propose the following conjecture.

Conjecture 2. *The number $w^*(3,3)$ exists. The numbers $w^*(4,3)$ and $w^*(2,4)$ do not exist.*

Our guess would be that $w^*(3,3) < 500$. Also we recall that $w^*(2,3) = 17$ and $w^*(4,2) \geq 30830$.

4 Double 3-term Arithmetic Progressions in Increasing Sequences of Positive Integers

In this section, we return to Problem 1: the existence of double 3-term arithmetic progressions in infinite sequences of positive integers with bounded gaps.

We remind the reader of the meaning of the following terms from the combinatorics of words.

		Max Green Gaps				
		3	4	5	6	7+
Max Blue Gaps	3	22				
	4	31	31			
	5	33	38	43		
	6	33	41	44	45	
	7	33	41	46	46	46
	8+	33	41	46	46	47

Max Red Gap 3

		Max Green			
		5	6	7	8+
Max Blue	5	100			
	6	> 113	> 133		
	7	?	?	?	
	8+	?	?	?	?

Max Red Gap 5

		Max Green Gaps					
		4	5	6	7	8	9+
Max Blue Gaps	4	39					
	5	49	63				
	6	56	79	91			
	7	76	96	>105	>121		
	8	81	96	>114	>131	>131	
	9	81	96	>114	>133	>133	>133
	10	81	96	>114	>133	>135	>135
	11+	81	97	>114	>133	>135	>135

Max Red Gap 4

Table 2: Known Values and Bounds for $w^*(3; a, b, c)$

An infinite word on a finite subset S of $\mathbb{Z}$, called the *alphabet*, is defined as a map $\omega : \mathbb{N} \to S$ and is usually written as $\omega = x_1 x_2 \cdots$, with $x_i \in S$, $i \in \mathbb{N}$. For $n \in \mathbb{N}$, a *factor* B of the infinite word ω of length $n = |B|$ is the image of a set of n consecutive positive integers by ω, $B = \omega(\{i, i+1, \ldots, i+n-1\}) = x_i x_{i+1} \cdots x_{i+n-1}$. The *sum* of the factor B is $\sum B = x_i + x_{i+1} + \cdots + x_{i+n-1}$. A factor $B = \omega(\{1, 2, \ldots, n\}) = x_1 x_2 \cdots x_n$ is called a *prefix* of ω.

Theorem 4. *The following statements are equivalent:*

(1) For all $k > 1$, every infinite word on $\{1, 2, ..., k\}$ has two adjacent factors with equal length and equal sum.

(1a) For all $k > 1$, there exists $R = R(k)$ such that every word on $\{1, 2, ..., k\}$ of length R has two adjacent factors with equal length and equal sum.

(2) For all $n > 1$, if $x_1 < x_2 < x_3 < \cdots$ is an infinite sequence of positive integers such that $x_{i+1} - x_i \leq n$ for all $i > 1$, then there exist $1 \leq i < j < k$ such that $x_i + x_k = 2x_j$ and $i + k = 2j$.

		Red	
		2	3
Blue	2	7	
	3	11	17

Double 3-AP's

		Red		
		2	3	4+
Blue	2	11		
	3	22	> 176	
	4+	22	> 2690	> 3573

Double 4-AP's

		Red			
		2	3	4	5+
Blue	2	15			
	3	37	> 131000		
	4	> 25503	?	?	
	5+	> 33366	?	?	?

Double 5-AP's

Table 3: Known Values and Bounds for $w^*(k; a, b)$

(2a) *For all $n > 1$, there exists $S = S(n)$ such that if $x_1 < x_2 < x_3 < \ldots < x_S$ are positive integers with $x_{i+1} - x_i \leq n$ whenever $1 \leq i \leq S-1$, then there exist $1 \leq i < j < k \leq S$ such that $x_i + x_k = 2x_j$ and $i + k = 2j$.*

(3) *For all $t > 1$, if $\mathbb{N} = A_1 \cup A_2 \cup ... \cup A_t$, then there exists q, $1 \leq q \leq t$, such that if $A_q = \{x_1 < x_2 < ...\}$, then there are $1 \leq i < j < k$ such that $x_i + x_k = 2x_j$ and $i + k = 2j$.*

(3a) *For all $t > 1$, there exists $T = T(t)$ such that for all $a > 1$, if $\{a, a + 1, ..., a + T - 1\} = A_1 \cup A_2 \cup ... \cup A_t$, then there exists q, $1 \leq q \leq t$, such that if $A_q = \{x_1 < x_2 < ... < x_p\}$, then there are $1 \leq i < j < k$ such that $x_1 + x_k = 2x_j$ and $i + k = 2j$.*

Remark 1. *Note that in (3) and (3a), we make statements about coverings, and not about partitions (colourings). This turns out to be essential, since if we used colourings in (3) and (3a) (call these new statements (3') and (3a')), then (3') would not imply (2), although (2) would still imply (3a'). This can be seen from the proofs below.*

Remark 2. *In each case $i = 1, 2, 3$, the statement (ia) is the finite form of the statement (i).*

Proof. We start by proving that (2) implies (2a). (The proof that (1) implies (1a) follows the same form, and is a little more routine.)

Suppose that (2a) is false. Then there exists n such that for all $S > 1$ there are $x_1 < x_2 < x_3 < ... < x_S$, with $x_{i+1} - x_i \leq n$ whenever $1 \leq i \leq S-1$, such that there do not exist $1 \leq i < j < k \leq S$ such that $x_i + x_k = 2x_j$ and $i + k = 2j$. Replace $x_1 < x_2 < x_3 < ... < x_S$ by its characteristic binary word (of length x_S)

$$B_S = b_1 b_2 b_3 ... b_{x_S}$$

defined by: $b_i = 1$ if i is in $\{x_1, x_2, x_3, ..., x_S\}$, and $b_i = 0$ otherwise. Let H be the (infinite) collection of binary words obtained in this way. Note that if B_S is in H, then each pair of consecutive 1s in B_S are separated by at most $n-1$ 0s.

Now construct inductively an infinite binary word w such that each prefix of w is a prefix of infinitely many words B_S in H in the following way. Let w_1 be a prefix of an infinite set H_1 of words in H. Let w_1w_2 be a prefix of an infinite set H_2 of words in H_1. And so on. Set $w = w_1w_2...$.

Define $x_1 < x_2 < x_3 < ...$ so that w is the characteristic word of $x_1 < x_2 < x_3 < ...$ and note that $x_{i+1} - x_i \leq n$ for all $i > 1$. Now it follows that there cannot exist $1 \leq i < j < k$ with $x_1 + x_k = 2x_j$ and $i + k = 2j$. (For these i, j, k would occur inside some prefix of w. But that prefix is itself a prefix of some word $B_S = b_1b_2b_3...b_S$, where there do not exist such i, j, k.) Thus if (2a) is false, (2) is false.

Next we prove that (3) implies (3a). Suppose that (3a) is false. Then there exists t such that for all T there is, without loss of generality, a covering $\{1, 2, ..., T\} = A_1 \cup A_2 \cup ... \cup A_t$ such that there does not exist q with $A_q = \{x_1 < x_2 < \cdots < x_p\}$ and $i < j < k$ with $x_1 + x_k = 2x_j$ and $i + k = 2j$. Represent the cover $\{1, 2, \ldots, T\} = A_1 \cup A_2 \cup ... \cup A_t$ by a word $B_T = b_1b_2b_3...b_T$ on the alphabet consisting of the non-empty subsets of $\{1, 2, ..., t\}$. Here for each i, $1 \leq i \leq T$, $b_i = \{$the set of $p, 1 \leq p \leq t,$ such that i is in $A_p\}$. Let H be the set of all words B_T obtained in this way. Construct an infinite word w, on the alphabet consisting of the non-empty subsets of $\{1, 2, \ldots, t\}$, such that each prefix of w is a prefix of infinitely many of the words B_T in H. Thus w represents a cover $\mathbb{N} = A_1 \cup A_2 \cup ... \cup A_t$, where $A_i = \{j \geq 1$ such that i is in $w_j\}$, $1 \leq i \leq t$, for which there does not exist i, $A_i = \{x_1 < x_2 < ...\}$, with $1 \leq i < j < k$ such that $x_1 + x_k = 2x_j$ and $i + k = 2j$, contradicting (3).

It is not difficult to show that (1) is equivalent to (2), that (1) is equivalent to (1a), that (2a) implies (2), and that (3a) implies (3). We have shown that (2) implies (2a) and that (3) implies (3a).

The final steps are:

Proof that (3) implies (2). If n and $A_0 = \{x_1 < x_2 < x_3 < ...\}$ are given, with $x_{i+1} - x_i \leq n$ for all $i > 1$, let $A_i = A_0 + i$, $0 \leq i \leq n-1$. Then $\mathbb{N} = A_0 \cup A_1 \cup ... \cup A_{n-1}$, and now (3) implies (2).

Proof that (2) implies (3a). Assume (2), and use induction on t to show that if $\mathbb{N} = A_0 \cup A_1 \cup ... \cup A_{t-1}$, then there exists q, $0 \le q \le t-1$, with $A_q = \{x_1 < x_2 < ...\}$ for which there exist $i < j < k$ with $x_i + x_k = 2x_j$ and $i + k = 2j$. For $t = 1$ this is trivial. Fix $t > 1$, assume the statement 3a for this t, and let $\mathbb{N} = A_0 \cup A_1 \cup ... \cup A_t$. If $A_t = \{x_1 < x_2 < ...\}$ is either finite or there exists n with $x_{i+1} - x_i \le n$ for all $i > 1$, then we are done by 2.

Otherwise, there are arbitrarily long intervals $[a, b] = B$ which are subsets of $A_0 \cup A_1 \cup ... \cup A_{t-1}$, and we are done by the induction hypothesis.

$\square$

Remark 3. *If true, perhaps (3a) can be proved by a method such as van der Waerden's proof that any finite colouring of $\mathbb{N}$ has a monochromatic 3-AP.*

Here is another remark on double 3-term arithmetic progressions.

Theorem 5. *The following two statements are equivalent:*

(1) For all $n \ge 1$, every infinite sequence of positive integers $x_1 < x_2 < \cdots$ such that $x_{i+1} - x_i \le n$ contains a double 3-term arithmetic progression.

(2) For all $n \ge 1$, every infinite sequence of positive integers $x_1 < x_2 < \cdots$ such that $x_{i+1} - x_i \le n$ contains a double 3-term arithmetic progression x_i, x_j, x_k with the property that $j - i = k - j \ge m$ for any fixed $m \in \mathbb{N}$.

Proof. Certainly (2) implies (1). We prove that (1) implies (2).

Let n and m be given positive integers. Let $X = \{x_1 < x_2 \cdots\}$ be an infinite sequence with gaps from $\{1, \ldots, n\}$. For $j \in \mathbb{N}$ we define $y_j = x_{jm+1} - x_{(j-1)m+1}$. Note that $m \le y_j \le nm$. Next we define an increasing sequence $Z = \{z_1 < z_2 < \ldots\}$ with gaps from $\{m, \ldots, nm\}$ by

$$z_i = \sum_{j=1}^{i} y_j = \sum_{j=1}^{i} x_{jm+1} - \sum_{j=0}^{i-1} x_{jm+1}.$$

By (1) the sequence Z contains a double 3-term arithmetic progression z_p, z_q, z_r with

$$z_r - z_q = z_q - z_p \text{ and } p + r = 2q.$$

It follows that

$$\sum_{j=q+1}^{r} x_{jm+1} - \sum_{j=q}^{r-1} x_{jm+1} = \sum_{j=p+1}^{q} x_{jm+1} - \sum_{j=p}^{q-1} x_{jm+1}$$

and

$$x_{rm+1} - x_{qm+1} = x_{qm+1} - x_{pm+1}.$$

From

$$(pm+1)+(rm+1) = m(p+r)+2 = 2mq+2 = 2(mq+1)$$

we conclude that $x_{pm+1}, x_{qm+1}, x_{rm+1}$ form a double 3-term arithmetic progression with gap

$$rm+1-(qm+1) = (r-q)m \geq m.$$

Since m and X are arbitrary, we conclude that (2) holds.

□

We wonder if one could get some intuitive "evidence" that it is easier to show that $w^*(3,3)$ exists than it is to show that every increasing sequence with gaps from $\{1,2,3,\ldots,17\}$ has a double 3-term arithmetic progression. The "17" is chosen because in a 3-colouring of $[1,m]$ which has no monochromatic double 3-AP, the gaps between elements of this colour class are coloured with 2 colours, and $w^*(2,3) = 17$.

`RamseyScript` was used for search of an increasing sequence with gaps from $\{1,2,3,\ldots,17\}$ with no double 3-term arithmetic progressions. The first search produced a sequence of the length 2207. The histogram with the distribution of gaps in this sequence is given on Figure 2.

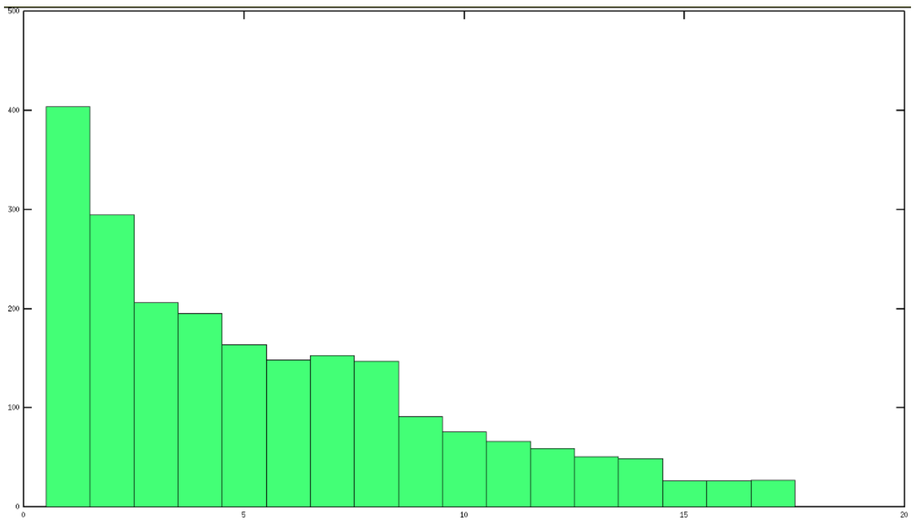

Figure 2: Histogram of Gaps in a 2207-term Double 3-AP Free Sequence

In another attempt we changed the order of gaps in the search, taking

$$[16, 12, 11, 17, 10, 14, 15, 8, 5, 3, 6, 4, 2, 1, 13, 7, 9]$$

instead of $[1,2,\ldots,17]$. `RamseyScript` produced a 5234-term double 3-AP free sequence. The corresponding histogram of gaps is given on Figure 3.

Here are a few conclusion that one can make from this experiment.

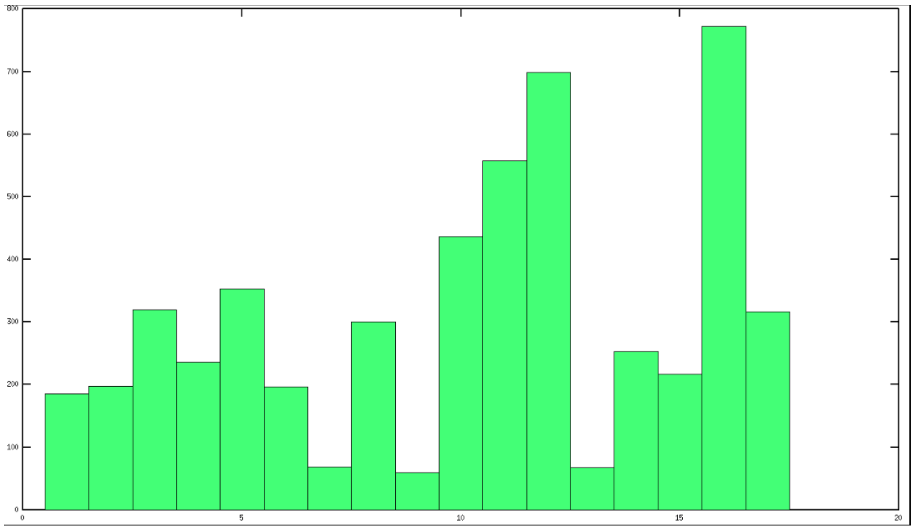

Figure 3: Histogram of Gaps in a 5234-term Double 3-AP Free Sequence

1. Initial choices of the order of gaps matter very much when constructing a double 3-AP free sequence, because we cannot backtrack in a reasonable (human) timespan at these lengths.

2. We do not really know anything about how long a sequence there will be.

3. The search space is very big. Table 4 gives the recursion tree size vs. maximum sequences considered.

Max. Sequence	Size of Search
0	1
1	18
2	307
3	4931
4	78915
5	1216147
6	18695275
7	278661995
8	????

Table 4: Recursion Tree Size vs. Maximum Sequences

5 RamseyScript

To handle the sheer number and variety of computations done for this project and related ones, we use the utility `RamseyScript`, a high-level scripting language developed by the third author. Originally created in December 2011, `RamseyScript` is, to the best of our knowledge, the first general-purpose computing tool for Ramsey Theory. After more than one year of development and over 200 changes, it is a mature, well-documented, indispensable tool. In creating `RamseyScript`, we had two goals:

- To define a precise and easily readable language for describing computational Ramsey theory problems.

- To provide an efficient means to actually carry out these computations.

To achieve these goals, `RamseyScript` is implemented as an ANSI C command-line program which can be used interactively or to read script files. Its own code is written to be highly modular, to ease both verification and extensibility. This allows users to communicate problems easily by email, keep results associated with the scripts that produced them, and quickly explore new problems. Also, users avoid writing opaque, error prone, throwaway C code, even when doing one-off computations.

At a very high level, `RamseyScript` builds a recursion tree given some restraints, and outputs a set of information about it. Since any problem of the form "what is the largest object satisfying the given condition?" can be cast in this way, this single behavior handles a much more diverse set of problems than it might appear. In fact, every numerical result and plot[2] in this paper was created with `RamseyScript`.

`RamseyScript` works with three main abstractions: search spaces, filters and targets.

The *search space* is the space of objects to be recursively generated. Each run of `RamseyScript` uses exactly one search space, for example `colorings` or `sequences`. The program starts from a chosen seed in the search space (e.g. the empty sequence) and tries to extend it without violating any set filters.

After each extension, `RamseyScript` checks what *filters* have been selected for the program run. If a filter fails, the current object is rejected along with all its extensions. This has the effect of pruning the recursion tree.

If all filters have passed, `RamseyScript` checks its list of *targets*. The default target, `max-length`, simply checks whether the current object is longer than the

[2]`RamseyScript` does not create plots directly, but outputs data suitable for processing by a different tool.

known maximum — if so, it stores the object to be output at the end of the program run.

Here is an example script to demonstrate these ideas and syntax:

```
# Output a brief description
echo Find the longest interval [1, n] that cannot be 4-colored
echo without a monochromatic 3-AP or a rainbow 4-AP.

# Set up environment
set n-colors 4
set ap-length 3

# Choose filters
filter no-n-aps
filter no-rainbow-aps

# Use the default target (max-length)

# Choose a search space and go!
search colorings
```

Of course, `RamseyScript` supports many other options to control output, run time, recursion depth, etc. The `RamseyScript` code is available to download at `https://www.github.com/apoelstra/RamseyScript`. It is licensed under the Creative Commons 0 public domain dedication license. For full details see the `README`.

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